Reliability of Two-Stage Weighted-q-out-of-*n* Systems with Components in Common

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Abstract

This paper extends the existing weighted-*k*-out-of-*n* model to two-stage weighted-**q**-out-of-*n* models with components in common. This model can be potentially applied in reliability study related to manufacturing and network reliability problems. Algorithms are developed to calculate the system reliability and generate the minimal cuts and minimal paths of two-stage weighted-**q**-out-of-*n* systems. Sensitivity analysis for the model parameters and reliability bounds with s-dependent component failures are also investigated based on the generated minimal cuts and minimal paths. Examples are provided to demonstrate the application of the developed models and algorithms.

Notation

$F(\mathbf{z}, s)$	$=1-R(\mathbf{z}, s)$
m	number of failure conditions used to define system failures of a two-stage weighted-**q**-out-of-*n* system
n	number of components in the system
$R(\mathbf{z}, s)$	reliability of a two-stage weighted-**z**-out-of-*s*:F system; therefore $R(\boldsymbol{q}, n)$ is reliability of the two-stage weighted-$\boldsymbol{q}$-out-of-*n*:F system
$R(q, n)$	reliability of a (one-stage) weighted-*q*-out-of-*n*:F system
p_i	probability that component *i* is working
PW-**q**-out-of-*n*	parallel-weighted-**q**-out-of-*n*
SW-**q**-out-of-*n*	series-weighted-**q**-out-of-*n*
$w_{ij} \geq 0$	The j^{th} component's weight assigned for the i^{th} failure condition for a two-stage weighted-**q**-out-of-*n* model, i=1, …, m, j=1, …, n
$\mathbf{w}_j$	$\equiv \begin{bmatrix} w_{1j} & \dots & w_{mj} \end{bmatrix}^T$, j=1, …, n
W_i	$= \sum_{j=1}^{n} w_{ij}$
$\boldsymbol{W}$	$\equiv \begin{bmatrix} W_1 & \dots & W_m \end{bmatrix}^T$
$\mathbf{x}$	$\equiv[x_1, \dots, x_n]^T$, where x_i is the state of component *i*: x_i=1 if component *i* is working; x_i=0 if it is failed
$\bar{\mathbf{x}}$	$\equiv[\bar{x}_1 \quad ... \quad \bar{x}_n]$, where $\bar{x}_i = 1 - x_i$

z $\equiv[z_1, \ldots, z_m]^T$

q $\equiv[q_1, \ldots, q_m]^T$

α_i, i=1, 2, …,m $\equiv \begin{cases} 1, & \text{if } \sum_{j=1}^{n} w_{ij}\bar{x}_j < q_i \\ 0 & \text{if } \sum_{j=1}^{n} w_{ij}\bar{x}_j \geq q_i \end{cases}$

α $\equiv[\alpha_1, \alpha_2, \ldots, \alpha_m]^T$

$\phi(\cdot)$ first-level structure function of a two-stage weighted-**q**-out-of-*n* model

1. Introduction

The weighted-*k*-out-of-*n* model is first proposed by Wu and Chen (1994) as an extension of the widely used *k*-out-of-*n* model. For a weighted-*k*-out-of-*n*:F system, a weight $w_i > 0$, which can be non-integers, is assigned to component *i*, *i*=1, …, *n*. The system is failed if and only if the total weight of the failed components is at least *k*, a pre-specified threshold value. In this paper, we replace *k* with the notation *q* since it can also be a non-integer. For a weighted-*q*-out-of-*n*:F system, the system is failed if and only if the total weight of the failed components is at least *q*. In this paper, the (one-stage) weighted-*q*-out-of-*n* model is extended to a two-stage weighted-**q**-out-of-*n* model with components in common. Examples of potential applications of the two-stage weighted-**q**-out-of-*n* models will be also suggested in the paper.

Let $F(z, j)$ represent the probability that a system with *j* components can output a total weight of at least *z*. Then the reliability of a weighted-*q*-out-of-*n* system can be obtained as $R(q,n) = 1 - F(q,n)$. Similarly to the result provided by (Wu and Chen 1994) for the weighted-*k*-out-of-*n*:G system, the reliability of the weighted-*q*-out-of-*n*:F system when all component failures are mutually *s*-independent can be obtained based on the following recursive relationship:

$$F(z, j) = \begin{cases} 1 & \text{for } z \leq 0,\ j \geq 0, \\ 0 & \text{for } z > 0,\ j = 0, \\ (1-p_j)F(z-w_j, j-1) + p_j F(z, j-1) & \text{otherwise} \end{cases} \tag{1}$$

where w_j denotes the weight of component *j*, *j*=1, 2, …, *n*.

2. Two-Stage Weighted-q-out-of-*n* Model with Components in Common

For a two-stage weighted-**q**-out-of-*n*:F model, the system is functioning if and only if $\phi(\boldsymbol{\alpha}) = 1$. Therefore, the two-stage weighted-**q**-out-of-*n* system can be decomposed into two hierarchical levels: the first (higher) level can be any coherent system, whose structure function is denoted by $\phi(\cdot)$; and the second (lower) level has a weighted-*q*-out-of-*n* structure. Special cases of two-stage weighted-**q**-out-of-*n* model include SW-**q**-out-of-*n* systems, whose first level has a series structure, and PW-**q**-out-of-*n* systems, whose first level has a parallel structure.

For example, a SW-**q**-out-of-*n*:F system is failed if and only if *any* of the inequalities in (2) is satisfied:

$$\begin{aligned} &w_{11}\bar{x}_1 + w_{12}\bar{x}_2 + \ldots + w_{1n}\bar{x}_n \geq q_1 \\ &\ldots \\ &w_{m1}\bar{x}_1 + w_{m2}\bar{x}_2 + \ldots + w_{mn}\bar{x}_n \geq q_m \end{aligned} \tag{2}$$

A PW-**q**-out-of-*n*:F system is failed if and only if *all* of the inequalities in (2) is satisfied. If the first level has a *k*-out-of-*m*:F structure, then the system is failed if and only if at least *k* of the *m* inequalities in (2) are satisfied. When *m* is equal to 1, the two-stage weighted-**q**-out-of-*n* system becomes a (one-stage) weighted-*q*-out-of-*n* system.

It should be noted there is a significant difference between the two-stage model in this paper and most of the two-stage models in the literature. Most two-stage systems studied in the literature are systems consisting of a number of *separate* identically structured modules. The modules are connected in a series structure, parallel structure, or more complicated structures. Usually, it is assumed that there are *no components in common* between any two modules. With this assumption, the failures of the modules are independent if all components have independent failures. And the system reliability can be calculated easily based on the reliability of each individual module. However, in the two-stage weighted-**q**-out-of-*n* model, failure conditions (inequalities in (2)) generally have *many* (*or all if all the weights* w_{ij} *are nonzero*) *components in common*. As a result, the system reliability cannot be evaluated by combining the failure probability corresponding to each failure condition in a straightforward way.

The extension of the one-stage G system is the two-stage weighted-**q**-out-of-*n*:G system. For a two-stage weighted-**q**-out-of-*n*:G system, the system is functioning if $\phi(\boldsymbol{\alpha}')=1$, where $\boldsymbol{\alpha}' \equiv [\alpha_1' \quad \dots \quad \alpha_n']^T$ with $\alpha_i' \equiv \begin{cases} 0, & \text{if } \sum_{j=1}^{n} w_{ij} x_j < q_i \\ 1 & \text{if } \sum_{j=1}^{n} w_{ij} x_j \ge q_i \end{cases}$. It can be seen that the two-stage weighted-***q***-out-of-*n*:F system is equivalent to a two-stage weighted-(**W**-**q**+ε)-out-of-*n*:G system with the same first level structure function ϕ, where $\boldsymbol{\varepsilon} \equiv [\varepsilon \quad \dots \quad \varepsilon]^T$ is an *m*×1 vector with $\varepsilon > 0$ being a very small number—smaller than the minimal unit of the component weights w_{ij} and the weight limit q_i, *i*=1, …, *m*, *j*=1, …, *n*. The usage of **ε** is to ensure that $\sum_{j=1}^{n} w_{ij}\bar{x}_j < q_i \Leftrightarrow \sum_{j=1}^{n} w_{ij}\bar{x}_j \le q_i - \varepsilon$, for all *i*=1, …, *m*. Therefore, all the results on the F system can be applied for the G system.

The dual structure is defined in Barlow and Proschan (1981) as

$$\phi^D(\mathbf{x}) = 1 - \phi(\mathbf{1} - \mathbf{x}).$$

By definition, it is not difficult to see that the two-stage weighted-**q**-out-of-*n*:F system with the first-level structure function $\phi(\cdot)$ and the two-stage weighted-**q**-out-of-*n*:G system with the first-level structure function $\phi^D(\cdot)$ are duals of each other.

3. Algorithms for System Reliability Evaluation and Generation of Minimal Cuts and Minimal Paths

In this section, all the component states are assumed as mutually *s*-independent. Reliability evaluation and generation of minimal paths and minimal cuts are first investigated for SW-**q**-out-of-*n*:F and PW-**q**-out-of-*n*:F systems. Then the result will be generalized for all two-stage weighted-**q**-out-of-*n* systems.

3.1 SW-q-out-of-*n*:F and and PW-q-out-of-*n*:F systems

The following recursive relationship can be used to calculate $R(\mathbf{q}, n)$ for a SW-**q**-out-of-*n*:F system, which is equal to $1-F(\mathbf{q}, n)$:

$$F(\mathbf{z}, j)=\begin{cases} 1 & \text{if } j \geq 0 \text{ and there exists } i, \text{s.t. } z_i \leq 0 \\ 0 & j=0 \text{ and } z_i > 0, \text{for all } i \\ (1-p_j)F(\mathbf{z}-\mathbf{w}_j, j-1)+p_j F(\mathbf{z}, j-1) & \text{otherwise} \end{cases} \tag{3}$$

The system reliability for a PW-**q**-out-of-*n*:F system can be calculated by:

$$F(\mathbf{z}, j)=\begin{cases} 1 & \text{if } j \geq 0 \text{ and } z_i \leq 0, \text{for all } i \\ 0 & j=0 \text{ and there exists } i, \text{s.t. } z_i > 0 \\ (1-p_j)F(\mathbf{z}-\mathbf{w}_j, j-1)+p_j F(\mathbf{z}, j-1) & \text{otherwise} \end{cases} \tag{4}$$

A recursive relationship can be obtained to generate minimal cuts of SW-**q**-out-of-*n*:F system and PW-**q**-out-of-*n*:F system. The minimal paths can be obtained as the minimal cuts of their dual systems.

3.2 General Two-Stage weighted-q-out-of-*n*:F Systems

From the recursive relationship in (3) and (4) for the SW-**q**-out-of-*n*:F system and the PW-**q**-out-of-*n*:F system, respectively, it can be seen that the only difference between these two equations is on the boundary conditions. Generally for a two-stage weighted-**q**-out-of-*n*:F system with first-level structure function $\phi(\cdot)$, the system reliability can be calculated based on the following recursive relationship:

$$F(\mathbf{z}, j)=\begin{cases} 1 & \text{if } j \geq 0 \text{ and } \phi(\boldsymbol{\alpha}_0)=0 \\ 0 & j=0 \text{ and } \phi(\boldsymbol{\alpha}_0)=1 \\ (1-p_j)F(\mathbf{z}-\mathbf{w}_j, j-1)+p_j F(\mathbf{z}, j-1) & \text{otherwise} \end{cases},$$

where

$$\boldsymbol{\alpha}_0 \equiv [\alpha_{01} \quad \dots \quad \alpha_{0m}] \text{ with } \alpha_{0i}=\begin{cases} 1, & \text{if } z_i > 0 \\ 0, & \text{if } z_i \leq 0 \end{cases}.$$

4. Applications of the two-stage weighted-q-out-of-*n* model

The two-stage weighted-**q**-out-of-*n* model developed in this paper has great application potential in many complex reliability problems. This section will introduce two areas of applications of the two-stage weighted-**q**-out-of-*n* model.

References

Barlow, R. E., and F. Proschan. (1981). *Statistical Theory of Reliability and Life Testing: Probability Models*. Silver Spring, MD: TO BEGIN WITH.

Wu, J., and R. Chen. (1994). An Algorithm for Computing the Reliability of Weighted-*k*-out-of-*n* Systems. *IEEE Transactions on Reliability* 43 (2):327-328.